IMTC Division A Samples

Enjoy our sample problems!

Problem 1

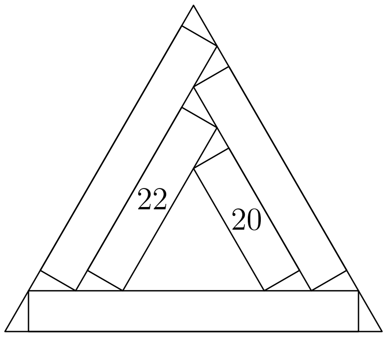
Let $P(x)$ be a quadratic polynomial such that the product of the roots of P is 20. Real numbers a and b satisfy $a+b=22$ and $P(a)+P(b)=P(22)$. Find a^2+b^2.

Problem 2

Find the unique three-digit prime number p with distinct digits $\underline{a}\ \underline{b}\ \underline{c}$ such that the last two digits of p^2 are $\underline{a}\ \underline{b}$ and the last digit of p^3 is $\underline{c}$.

Problem 3

Five rectangles with the same height are positioned in an equilateral triangle as shown. Two of the rectangles have areas 20 and 22, as indicated. Find the area of the triangle.



Problem 4

Given that the distinct real numbers r_1, r_2, and r_3 are roots of the equation

$$12r_1x^3 + 72r_2x^2 + 432r_3x = 0$$

what is the value of ${r_1}^2 + {r_2}^2 + {r_3}^2$?

Problem 5

Let $\mathcal{S}$ be the set of the first 8 prime numbers. For each subset $\mathcal{T}$ of $\mathcal{S}$, let $f(\mathcal{T})$ be the remainder when the product of the elements of $\mathcal{T}$ is divided by 6. Find the sum of $f(\mathcal{T})$ over all subsets of $\mathcal{S}$. Note: for the empty set, we define $f(\emptyset)=1$.

Problem 6

Let $ABCD$ be a rectangle and let ω be the circle with center A that passes through C. If line BD intersects ω at points P and Q so that $PB < PD$. Given that $PB = 10$ and $DQ = 12$, the area of $ABCD$ can be written as $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m+n$

Problem 7

Let $P(n)$ be a quadratic polynomial with integer coefficients. Jordan finds that, for all positive integers $n \geq 3$, every term of the sequence

$$P(n),\ P(P(n)),\ P(P(P(n))),\ \cdots$$

is a positive integer relatively prime to n. However, any two consecutive terms of this sequence sum to a multiple of n. Find $P(10)$.

Problem 8

Three points A, B, and C lie on the graph of $y = x^2$. Given that lines AB, BC, and CA contain points $(1, -7)$, $(1, 4)$, and $(1, 7)$ respectively, the area of ABC can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Find $m + n$.

Problem 9

Define a sequence of rational numbers, $a_0, a_1, a_2, \cdots$ so that $a_{k+1} = |1 - \frac{1}{a_k}|$. Find the number of possible values of a_0 so that $a_{12} = 0$.

Problem 10

Juan chooses a random divisor of 2310 and writes it on a blackboard. He repeats this process indefinitely, and stops when two numbers on the board are not relatively prime. Let N be the number of ways in which Juan stops after 4 numbers are written on the board. Compute the remainder when N is divided by 1000

Problem 11

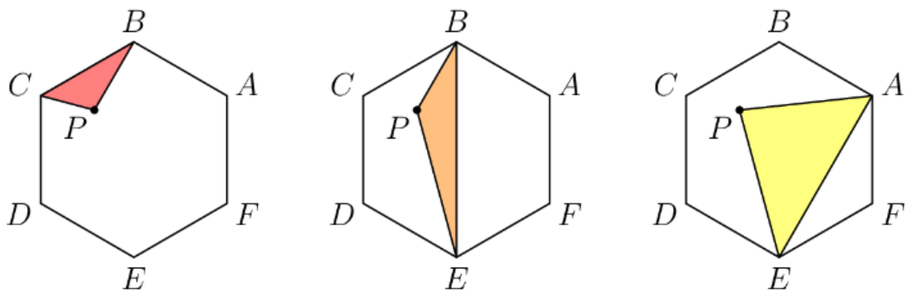
Jessica is writing a sequence a_i on a whiteboard. For all positive integers i, she defines

$$a_i = \sqrt{\frac{i + 100}{i}}.$$

Let N be the sum of all i for which a_i is a rational number. Compute the remainder when N is divided by 1000.

Problem 12

A point P is positioned inside regular hexagon $ABCDEF$ so that $CP < AP$. Triangles BPC, BPE, and APE have areas 7, 12, and 28, respectively. Find the area of the hexagon.



Problem 13

There exist quadratic functions $P(x)$ and $Q(x)$ with real coefficients. Each of the three equations

$$P(x) = Q(x)$$

$$P(x) = 2Q(x)$$

$$P(x) = 3Q(x)$$

has exactly 1 solution. Given that the solutions to these three equations are $x = 20$, 22, and n for a real number n, find the remainder when the product of all possible values of n is divided by 1000.

Problem 14

Paul has an ample number of building blocks with identical bases and heights of 1, 2, 3, 4, 5. With exactly 9 of these blocks, let m be the number of combinations of 2 distinct towers of equal height so that the bases the 7 blocks not touching the ground each have a unique elevation. Find the remainder then m is divided by 1000. (Note: blocks of the same height are identical and the order of towers doesn't matter)

Problem 15

Let ABC be an isosceles triangle with $AB = BC = 70$. Circles ω_1 and ω_2 are externally tangent at point T, and are tangent to AB and BC at A and C respectively. The center of ω_1 is equidistant from B and C. The median from C to AB intersects ω_2 at $E \neq C$. Given that $AT = 52$, find EC.