IMTC High School

Rules

This is a fifteen question short answer exam. Each question has a positive integer answer. Questions are not necessarily ordered by difficulty, and diagrams are not necessarily drawn to scale. You will have have 75 minutes to finish the exam. For the first 10 questions, you will receive 6 points for a correct answer and 0 points for an incorrect answer or if the question is left blank. For the last 5 questions, you will receive 8 points for a correct answer and 0 points for an incorrect answer or if the question is left blank. No aids are permitted other than scratch paper, graph paper, rulers, and writing instruments. The use calculators, smartwatches, or computing devices is prohibited. No problems on the exam will require the use of a calculator. Discussion about problems from this exam before submission and before the submission deadline is strictly prohibited.

Do not exit the tab or switch to another tab at any point during the exam. Doing so lowers the integrity of the exam and might result in a disqualification.

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WOLFRAM

Problem 1

Let A and B be digits such that the two digit integer $\overline{AB}$ is divisible by 8 and the two digit integer $\overline{BA}$ is divisible by 23. Find the maximum possible value of $A^2 + B^2$.

Problem 2

For a positive integer n, a student rolls n fair six-sided die. He notes that the sum of the rolls is 25 and the product of the rolls is 225. What is the maximum possible value of n?

Problem 3

For all real numbers a such that $-10 \leq a \leq 10$, the graph of $(x-a)^2 + (y-a)^2 \leq a^2$ is drawn on a coordinate plane. If the area of the region contained in the union of these graphs can be represented as $p + q\pi$ for integers p and q, find $p + q$.

Problem 4

Harry has an ample number of identical red marbles and green marbles. How many ways can he grab 5 of these marbles and arrange them in a line such that there is at least one red marble in the line that does not neighbor another red marble?

Problem 5

Let m and n be positive integers such that

$$\frac{\frac{1}{2} + \log_{36}\left(\sqrt{\frac{m}{n}}\right)}{\log_6 m} = \frac{1}{2}.$$

Find the sum of the distinct possible values of $m + n$.

Problem 6

Let a permutation of the set $1, 2, 3, \cdots 9$ be $P = a_1, a_2 \ldots a_9$ where a_i is the ith term of the permutation. Find the number of permutations P that exist such that

$$(a_1+a_2+a_3) < (a_2+a_3+a_4) < (a_3+a_4+a_5) < \cdots < (a_7+a_8+a_9).$$

Problem 7

Find the sum of all two digit positive integers n such that the following statement is true:

$$n \cdot \gcd(n, 21) \equiv 21 \bmod 41.$$

Problem 8

In triangle $\triangle ABC$, D, E, and F lie on sides AC, AB and BC respectively such that $DF \perp BC, EF \perp AB$, and $ED \perp AD$. Given that $EF = \frac{BF}{2} = \frac{DF}{3} = 10$, determine the area of $\triangle DEF$.

Problem 9

Given that the distinct real numbers, r_1, r_2 and r_3 are roots of the equation,

$$12r_1x^3 + 72r_2x^2 + 432r_3x = 0,$$

what is the value of $r_1^2 + r_2^2 + r_3^2$?

Problem 10

Let $ABCD$ be a isosceles trapezoid such that $AB < CD$ and let and let E be the orthocenter of $\triangle BCD$. Given that $\triangle EAD$ is a equilateral triangle with side length 12, and the area of trapezoid $ABCD$ can be expressed as $a + b\sqrt{c}$, determine $a + b + c$.

Problem 11

For real numbers $a > 3b > 0$, let the maximum value of

$$\frac{3a + 8a^2b - 16ab^2}{2b^2 - ab}$$

be a real number, m. Determine m^2.

Problem 12

How many positive integers $n \leq 242$ exist such that the remainders when n is divided by $1,\ 2,\ 11,\ 22,\ 121$ and 242 respectively form a non-decreasing sequence?

Problem 13

Joey draws three random not necessarily disjoint arcs on a unit circle. The probability that all points on the circumference of the circle is covered by at least one of the arcs can be represented in simplest form as $\frac{a}{b}$. Compute $a+b$.

Problem 14

Let $\triangle ABC$ be a triangle with $\angle A = 45^\circ$. Also, let S be a square inscribed in $\triangle ABC$ such that 2 vertices of S lie on BC and the other 2 vertices lie on AB and AC respectively. Let D be the center of square S and let ray AD intersect BC at point E. If $BE = 2$, and $CE = 3$, the area of triangle $\triangle ABC$ can be expressed as $\frac{a+b\sqrt{c}}{d}$ Find $a+b+c+d$.

Problem 15

Paul has an ample number of building blocks with identical bases and heights of 1, 2, 3, 4, 5. With exactly 9 of these blocks, how many ways can Paul build 2 distinct towers of the same height such that the bases of the 7 blocks that don't touch the ground have a unique elevation? (Note: The towers were built on the same flat surface)