IMTC Middle School

Rules

This is a fifteen question short answer exam. Each question has a positive integer answer. Questions are not necessarily ordered by difficulty, and diagrams are not necessarily drawn to scale. You will have have 75 minutes to finish the exam. For the first 10 questions, you will receive 6 points for a correct answer and 0 points for an incorrect answer or if the question is left blank. For the last 5 questions, you will receive 8 points for a correct answer and 0 points for an incorrect answer or if the question is left blank. No aids are permitted other than scratch paper, graph paper, rulers, and writing instruments. The use calculators, smartwatches, or computing devices is prohibited. No problems on the exam will require the use of a calculator. Discussion about problems from this exam before submission and before the submission deadline is strictly prohibited.

Do not exit the tab or switch to another tab at any point during the exam. Doing so lowers the integrity of the exam and might result in a disqualification.

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WOLFRAM

Problem 1

Given that $20^2 + 21^2 + a$ is the square of an integer, determine the least possible positive value of a.

Problem 2

Let there exist three not necessarily disjoint regions with areas $36, 16,$ and 4. Given that the area of the intersection of each pair of two regions, is at most one-fourth of the area of the smaller region, find is the minimum possible area that can be covered by the union of all three regions.

Problem 3

Let A and B be digits such that the two digit integer $\overline{AB}$ is divisible by 8 and the two digit integer $\overline{BA}$ is divisible by 23. Find the maximum possible value of $A^2 + B^2$.

Problem 4

For all real numbers a such that $-10 \leq a \leq 10$, the graph of $(x-a)^2 + (y-a)^2 \leq a^2$ is drawn on a coordinate plane. If the area of the region contained in the union of these graphs can be represented as $p + q\pi$ for integers p and q, find $p+q$.

Problem 5

Right triangle ABC with the right angle at B is drawn. Altitude BX is drawn to side AC. Point Y is chosen outside of $\triangle ABC$ such that $BXYC$ is an isosceles trapezoid, with $BC||XY$ and $BX = CY$. If $AB = 12$ and $AC = 24$, find XY^2.

Problem 6

Harry has an ample number of identical red marbles and green marbles. How many ways can he grab 5 of these marbles and arrange them in a line such that there is at least one red marble in the line that does not neighbor another red marble?

Problem 7

Find the sum of all two digit positive integers n such that the following statement is true:

$$n \cdot \gcd(n, 21) \equiv 21 \bmod 41$$

.

Problem 8

Let a permutation of the set $1, 2, 3, \cdots 9$ be $P = a_1, a_2, \cdots, a_9$ where a_i is the ith term of the permutation. Find the number of permutations P that exist such that

$$(a_1+a_2+a_3) < (a_2+a_3+a_4) < (a_3+a_4+a_5) < \cdots < (a_7+a_8+a_9).$$

Problem 9

In triangle $\triangle ABC$, D, E, and F lie on sides AC, AB and BC respectively such that $DF \perp BC, EF \perp AB$, and $ED \perp AD$. Given that $EF = \frac{BF}{2} = \frac{DF}{3} = 10$, determine the area of $\triangle DEF$.

Problem 10

Given that the distinct real numbers, r_1, r_2 and r_3 are roots of the equation,

$$12r_1x^3 + 72r_2x^2 + 432r_3x = 0,$$

what is the value of $r_1^2 + r_2^2 + r_3^2$?

Problem 11

Let $\triangle ABC$ be a triangle such that $AB = 13$, $BC = 14$, and $AC = 15$. Let P be a point in the same plane as $\triangle ABC$ such that the points A, B, C and P can be connected to form a not necessarily convex polygon where only $\angle P$ exceeds 90°. Find the area of the region that P can lie in.

Problem 12

For positive integers n greater than 1, let $f(n)$ be the largest prime factor of n. There exists an infinite recursive sequence, $a_0, a_1, a_2, a_3, \cdots$, such that $a_k + f(a_k) = a_{k+1}$ for all non-negative integers k. Given that $a_0 = 2$, determine the value of the smallest positive integer x such that the number a_x is divisible by 2022.

Problem 13

Define $a\#b$ to be $\frac{a+b}{a-b}$. Let the sum of positive integers $n \leq 2021$ such that $2021\#n$ is an integer be x. Determine $x \pmod{1000}$.

Problem 14

Let $ABCD$ be a isosceles trapezoid such that $AB < CD$ and let and let E be the orthocenter of $\triangle BCD$. Given that $\triangle EAD$ is a equilateral triangle with side length 12, and the area of quadrilateral $ABCD$ can be expressed as $a + b\sqrt{c}$, determine $a + b + c$.

Problem 15

Joe and Moe are playing a game where Moe initially selects a random integer from 1 to 513 inclusive. Then, Joe repeatedly guesses integers with a goal of knowing Moe's number. After each of Joe's guesses, Moe will respond in the following way:

- If Joe's guess is strictly less than Moe's number, Moe will say the phrase 'guess higher'.
- If Joe's guess is strictly greater than Moe's number, Moe will say the phrase 'guess lower'.
- If Joe's guess is equal to Moe's number, Moe will randomly say 'guess lower' or 'guess higher' with equal chance.

Joe guesses numbers in a way such that no matter what Moe's number is, the expected number of guesses Joe will take to guess that number is at most the real number x. If the smallest possible value of x can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n, find $m + n$.