Rules For The Individual Round

1. All calculators, both handheld and online are prohibited.
2. Pictures are not necessarily drawn to scale.
3. Discussion about problems from this test before submission, and before the test submission deadline is strictly prohibited.
4. There are 20 questions on this test with a time constraint of 75 minutes.
5. Questions are not necessarily ordered by difficulty.
6. You will be awarded 4 points for a correct answer, 0 points for a blank answer and 0 points for an incorrect answer.

Good Luck!

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IMTC 2020 Individual Round

The IMTC Team

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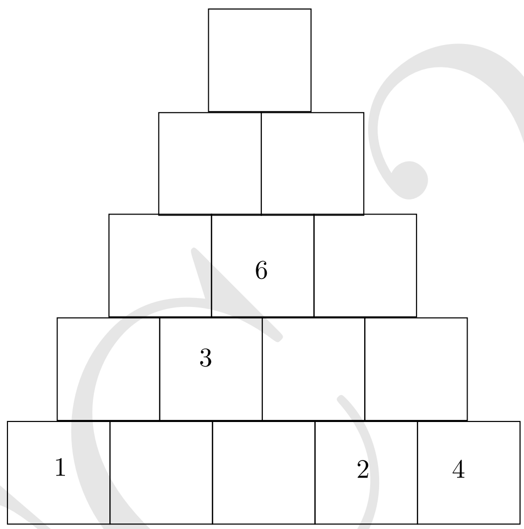
Problem I1

There exists a pyramid consisting of squares which either are blank or have a number in them as shown. Sally will write numbers in all the blank squares such that for any square not on the bottom row of the pyramid, the number in the square is equal to the sum of the two squares touching its bottom edge. Once Sally is done filling the squares in, find the number that will end up on the top square of the pyramid.



Problem I2

The positive factors of the positive integer 2^m are written down. If sum of these positive factors is 255, what is m?

Problem I3

If a_b represents 'a base b', For what value x does this statement hold true?

$$22_{2x} = 32_{x+1}$$

Problem I4

A student counts natural numbers. However, he skips multiples of 7. If a number he skips has factors other than 7 that are greater than 1, he also skips multiples of each of the other factors that are greater than 1 of that number, from that point onward. If he skips 5 numbers continuously, he stops counting. What is the last number the student counted?

The first few numbers he counts are:
$1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 15, 17$

Problem I5

How many strings of digits $\overline{abc}$, where when it is written in base 10 is less than 500, exist such that when string $\overline{abc}$ is written in base 16 and 7 such respectively, it has a base 10 equivalency of a square. For example, the string 100 would be valid because 100_{16} and 100_7 when written in base 10 are both perfect squares.

Problem I6

This question was stated incorrectly on the exam in a way that made it unsolvable and everyone received a point for it. The corrected version is shown below.

There exists a sequence $\{x_k\}$ with the first term being x_1, such that for all x_k in the sequence, $x_{k-1}(-1)^k + x_k = k$ for $k > 1$. Given that $x_1 + x_2 + x_3 + \cdots + x_m = 2550$ and m is even, find the value of m.

Problem I7

A triangle ABC is drawn. Angle bisector AX is drawn and a point Y is placed on AC such that $YX \perp BC$. YB intersects AX at point Z. If $\angle ABC = 90°$, $AB = 6$, and $AC = 10$, what is the area of triangle AZB? If the answer as a common fraction can be represented as $\frac{a}{b}$, find $a + b$.

Problem I8

An 'Expensive Number' is a number which can be expressed in the form $10001a + 1010b + 100c$ for non negative integers a, b and c such that $a, b, c \leq 9$. Find the sum of the sum of digits of all 'Expensive Numbers'.

Problem I9

Two secant lines, AX, AY pass through a circle with center O, and with A outside of the circle. AX intersects the circle at points C, D where C is closer to A. AY intersects the circle at points E, F where E is closer to A. $AC = 4, CD = 5, AE = 3$, and the perpendicular from O to EF has length $\frac{3\sqrt{3}}{2}$. If the area of the circle can be represented as $a\pi$, what is a?

Problem I10

The lines $y = 3x - 2, y = 5x + 6$, and $y = 2x$ are drawn on a coordinate plane. Let the intersections of these lines be A, B, and C. A new plane is constructed such that it intersects the original plane at a line $x = 5$ on the original plane, and such that it makes a 30 degree angle with the original plane. Now, from each of points A, B, C, a line is constructed perpendicular to the original plane and these lines intersect the new plane at points A', B', and C'. If the area of triangle $A'B'C'$ can be represented as $\frac{x\sqrt{y}}{z}$, what is xyz?

Problem I11

A magical unfair 3 sided dice with sides labeled 1, 2, and 3 is flipped 3 times. After these flips, the probability of rolling the number 1 exactly once is the same as the probability of rolling a 2 three times and is 8 times the probability of rolling a 3 three times. Given that for all three numbers, the probability it is rolled is constant, find the probability that in one roll, the number 1 is shown. If the answer can be represented as the common fraction $\frac{a}{b}$, find $a + b$.

Problem I12

Let $\lfloor x \rfloor$ represent the largest integer less than or equal to x and let $\lceil x \rceil$ represent the smallest integer greater than or equal to x. Find the number of ordered pairs of (a, b) where $a, b \in \mathbb{Z}$ such that $\lfloor \frac{a}{3} \rfloor \lceil \frac{b}{5} \rceil = 20^9$.

Problem I13

In triangle ABC, points X, Y, and N are placed on AB, AC, and BC respectively such that $\angle XNY = 90$. If $XY || BC$ and the lengths of XN, YN, XY, and BC form an arithmetic sequence with the common difference 5, find the area of triangle ABC.

Problem I14

On an infinite staircase with the lowest step having a height of 2 feet, each succeeding step has half the height of the previous step. Every second, a bug starting at the lowest step will jump to the step either one, two, or three above of it with equal probability. It is given that it starts on the lowest step and will do this process without turning. Let n be the expected value of the sum of the heights of the steps he jumped or walked on after getting to the top step. Find n. If the answer as a common fraction can be represented as $\frac{a}{b}$, find $a + b$.

Problem I15

On a true/false test, every answer is true or false. A student knows on a certain 10 question true or false test, there are an equal number of questions with both answers. He knows the answers to the first 5 questions, however has not looked at the last 5. If the student guesses all the remaining five questions such that his expected score is maximized, find the value of his expected score. If the answer as a common fraction can be represented as $\frac{a}{b}$, find $a + b$.

Problem I16

Each person in a line of 20 people is given a card with a different number between 1 and 20 written on it. For 4 people in the line, Alex, Bob, Carla, and Dan, the probability that the sum of the numbers Alex and Bob received is equal to the sum of the numbers Carla and Dan received can be expressed as the common fraction $\frac{a}{b}$. Find the sum of the distinct prime factors of ab.

Problem I17

In isosceles triangle ABC with $AB = AC$ and $AB > BC$, a circle with radius 2 is drawn such that it is internally tangent to AB and AC but not BC. From the center O of this circle, isosceles triangle OPQ with $OP = OQ$ is constructed such that PQ lies on BC and PQ has the same length as the diameter of the circle with center O. The circumcircle which has radius 3 of OPQ is constructed, and is tangent to AB and AC. What is the ratio of the areas of triangles ABC and OPQ?

If the answer can be represented in simplest form as $\frac{a\sqrt{b}+c\sqrt{d}}{e}$, what is $a + b + c + d + e$?

Problem I18

Kite $ABCD$ with $AB = AD$ and $CD = CB$ is drawn such that $BD = 8$. Let the intersection of AC and BD be O and $AO = 3$. A circle with center P is inscribed in this kite. A new circle with center U and the same radius as circle P is drawn such that it passes through P and is tangent to BD. If $PU \perp AD$, find the radius of circle P. If the answer as a common fraction can be represented as $\frac{a}{b}$, find $a + b$.

Problem I19

Toady the frog likes hopping and counting. He counts from the number 1 to 720 and if the number he says is relatively prime to his favorite number x, he hops forward 1 unit. If he says a number that is not relatively prime to x, he hops backward 1 unit. At the end of his counting, if he is 10 units away from his original starting point and x is as small as possible, what is Toady's favorite number?

Problem I20

A figure is constructed as follows using iterative steps:

- In the first step, a triangle is drawn.
- In the next step, a quadrilateral is constructed on each point of the triangle, such that no shapes overlap.
- Next, a pentagon is constructed on each point that is not shared by two points already, and such that no shapes overlap.
- In general, in the nth step, polygons with $n+2$ sides are drawn on each point that is not yet shared by two shapes, and in such a way that no shapes overlap.

Let $f(n)$ be the total number of sides of the figure after the nth step has been completed. Find $\left(\sum_{a=1}^{210} f(a)\right) \pmod{210}$