IMTC Division A

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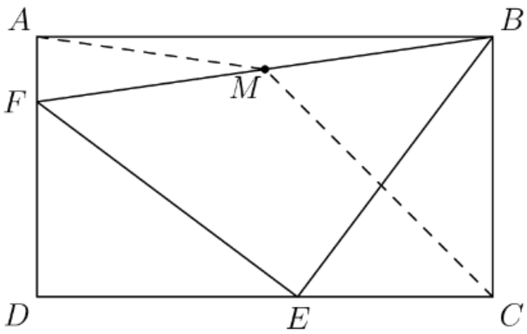
Problem 1

Let $P(x)$ be a quadratic polynomial such that the product of the roots of P is 20. Real numbers a and b satisfy $a + b = 22$ and $P(a) + P(b) = P(22)$. Find $a^2 + b^2$.

Problem 2

In rectangle $ABCD$, points E and F are chosen on sides CD and DA, respectively, such that triangle BEF is an isosceles right triangle with vertex E. Let M be the midpoint of BE. If $MA = 5$ and $MC = 7$, find the area of $ABCD$.



Problem 3

Let x and y be real numbers satisfying the system

$$2^x - 81y^2 = 0$$
$$3^x - 512y^3 = 0$$

Then y can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 4

Let $\mathcal{S}$ be the set of the first 8 prime numbers. For each subset $\mathcal{T}$ of $\mathcal{S}$, let $f(\mathcal{T})$ be the remainder when the product of the elements of $\mathcal{T}$ is divided by 6. Find the sum of $f(\mathcal{T})$ over all subsets of $\mathcal{S}$. Note: for the empty set, we define $f(\emptyset) = 1$.

Problem 5

Three distinct lattice points in the Cartesian plane form a *set* if their x coordinates are either all the same or all different, and their y coordinates are either all the same or all different. Find the number of combinations of 4 distinct lattice points with x and y coordinates between 1 and 3, inclusive, such that no 3 of them form a *set*.

Problem 6

Let $ABCD$ be a rectangle and let ω be the circle with center A that passes through C. If line BD intersects ω at points P and Q so that $PB < PD$. Given that $PB = 10$ and $DQ = 12$, the area of $ABCD$ can be written as $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

Problem 7

Let $P(n)$ be a quadratic polynomial with integer coefficients. Jordan finds that, for all positive integers $n \geq 3$, every term of the sequence

$$P(n),\ P(P(n)),\ P(P(P(n))),\ \cdots$$

is a positive integer relatively prime to n. However, any two consecutive terms of this sequence sum to a multiple of n. Find $P(10)$.

Problem 8

For relatively prime positive integers x and y, define $f(x, y)$ to be the smallest positive multiple of x that is 1 more than a multiple of y. For relatively prime positive integers p and q,

$$f(p,\ q) + f(q,\ p) = 4321$$

Compute the remainder when the sum of all distinct possible values of $p + q$ is divided by 1000.

Problem 9

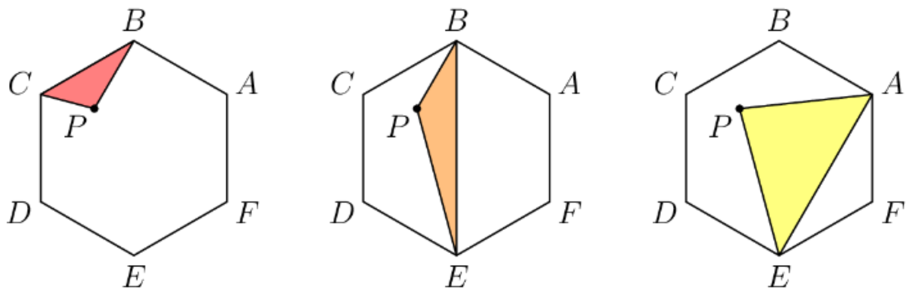
Define a sequence of rational numbers, $a_0, a_1, a_2, \cdots$ so that $a_{k+1} = |1 - \frac{1}{a_k}|$. Find the number of possible values of a_0 so that $a_{12} = 0$.

Problem 10

Juan chooses a random divisor of 2310 and writes it on a blackboard. He repeats this process indefinitely, and stops when two numbers on the board are not relatively prime. Let N be the number of ways in which Juan stops after 4 numbers are written on the board. Compute the remainder when N is divided by 1000.

Problem 11

A point P is positioned inside regular hexagon $ABCDEF$ so that $CP < AP$. Triangles BPC, BPE, and APE have areas 7, 12, and 28, respectively. Find the area of the hexagon.



Problem 12

There exist quadratic functions $P(x)$ and $Q(x)$ with real coefficients. Each of the three equations

$$P(x) = Q(x)$$
$$P(x) = 2Q(x)$$
$$P(x) = 3Q(x)$$

has exactly 1 solution. Given that the solutions to these three equations are $x = 20$, 22, and n for a real number n, find the remainder when the product of all possible values of n is divided by 1000.

Problem 13

All the vertices of the triangle ABC lie on the graph $y = \frac{1}{x}$ on a cartesian plane. If its orthocenter lies on the point $(-1, -1)$ and its centroid lies on the point $(7, 9)$, the length of its circumradius can be expressed as $\sqrt{m}$. Find m.

Problem 14

Bob has a shuffled deck of 12 cards each labelled a distinct integer from $1 - 12$. Every minute, he randomly chooses 2 from the deck, records their product on a board, and then removes them from the deck. After there are no more cards in the deck, the probability that all of the numbers on the Bob's board are at most 60 can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Find $m+n$.

Problem 15

Let ABC be an isosceles triangle with $AB = BC = 70$. Circles ω_1 and ω_2 are externally tangent at point T, and are tangent to AB and BC at A and C respectively. The center of ω_1 is equidistant from B and C. The median from C to AB intersects ω_2 at $E \neq C$. Given that $AT = 52$, find EC.