IMTC Division A Solutions

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Division A Answer Key

1	464	6	134	11	87
2	56	7	89	12	72
3	73	8	223 or 544	13	269
4	576	9	178	14	247
5	54	10	643	15	24

Problem 1

Let $P(x)$ be a quadratic polynomial such that the product of the roots of P is 20. Real numbers a and b satisfy $a+b=22$ and $P(a)+P(b)=P(22)$. Find a^2+b^2.

Proposed by Tanishq Pauskar

Solution by Govind Velamoor
By Vieta's, we know the polynomial must be of the form $px^2+qx+20p$. Therefore,

$$P(a)+P(b)=pa^2+qa+20p+pb^2+qb+20p=p(a^2+b^2)+q(a+b)+40p.$$

This is the same as

$$P(22)=22^2p+22q+20p,$$

so

$$p(a^2+b^2)+q(a+b)+40p=22^2p+22q+20p,$$
$$p(a^2+b^2)+q(a+b)+20p=22^2p+22q.$$

Because we know that $a+b=22$,

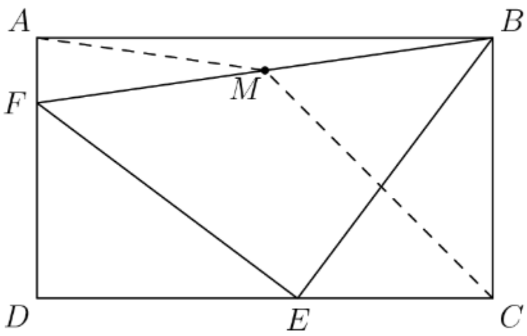
$$p(a^2+b^2)+22q+20p=22^2p+22q$$

$$p(a^2+b^2)+20p=22^2p$$
$$a^2+b^2+20=22^2.$$

So, we know $a^2+b^2=\boxed{464}$.

Problem 2

In rectangle $ABCD$, points E and F are chosen on sides CD and DA, respectively, such that triangle BEF is an isosceles right triangle with vertex E. Let M be the midpoint of BE. If $MA = 5$ and $MC = 7$, find the area of $ABCD$.



Proposed by Aarush Khare

Solution by Aarush Khare

First, notice that M is the circumcenter of $ABEF$, so $MA = MB = ME = MF = 5$, making $FE = EB = 5\sqrt{2}$. Then, note that the distance from M to DC is the average BC and FD, as M is the midpoint of BF.

Let $BC = a$ and $EC = b$. Notice that triangles FDE and ECB are congruent, so $DE = a$ and $FD = b$. Then the distance from M to DC is $\frac{a+b}{2}$, while the length of DC is $a + b$. This implies that triangle MDC is an isosceles right triangle, meaning $DC = 7\sqrt{2}$.

Thus, we have the system

$$a + b = 7\sqrt{2}$$
$$a^2 + b^2 = 50$$

which has solution $(a, b) = (4\sqrt{2}, 3\sqrt{2})$ (note that the opposite doesn't work as F lies outside the rectangle). Then the area of the rectanlge is $BC \cdot CD = 4\sqrt{2} \cdot 7\sqrt{2} = \boxed{56}$.

Problem 3

Let x and y be real numbers satisfying the system

$$2^x - 81y^2 = 0$$
$$3^x - 512y^3 = 0$$

Then y can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Proposed by Aarush Khare

Solution by Aarush Khare
Rearranging the equations, we get

$$2^x = 81y^2$$
$$3^x = 512y^3.$$

Cubing the first equation and squaring the second, we get

$$8^x = 3^{12}y^6$$
$$9^x = 2^{18}y^6.$$

Dividing these two yields

$$\left(\frac{8}{9}\right)^x = \frac{3^{12}}{2^{18}} = \frac{9^6}{8^6} = \left(\frac{8}{9}\right)^{-6}$$

so $x = -6$. Plugging this into the first equation yields

$$y^2 = \frac{1}{3^4 \cdot 2^6}$$

which implies that $y = \frac{1}{72}$, which yields an answer of $\boxed{73}$.

Problem 4

Let $\mathcal{S}$ be the set of the first 8 prime numbers. For each subset $\mathcal{T}$ of $\mathcal{S}$, let $f(\mathcal{T})$ be the remainder when the product of the elements of $\mathcal{T}$ is divided by 6. Find the sum of $f(\mathcal{T})$ over all subsets of $\mathcal{S}$. Note: for the empty set, we define $f(\emptyset) = 1$.

Proposed by Tanishq Pauskar, Govind Velamoor

Solution by Tanishq Pauskar
Consider a subset $\mathbb{T}$ where $f(\mathbb{T}) \neq 0$ that does not contain the element 5.
Let $\mathbb{P}$ be the union of $\mathbb{T}$ and $\{5\}$. It is easy to see that

$$f(\mathbb{T}) + f(\mathbb{P}) = 6$$

This means across all subsets $\mathbb{X}$ where $f(\mathbb{X}) \neq 0$, the expected value of $f(\mathbb{X})$ is 3. The remainder is not equal to 0 when either 2 or 3 are not included in the set (which happens $\frac{3}{4}$ of the time). There are a total of $2^8 = 256$ subsets of $\mathbb{S}$. Thus, our answer is

$$256 \cdot \frac{3}{4} \cdot 3 = \boxed{576}.$$

Problem 5

Three distinct lattice points in the Cartesian plane form a *set* if their x coordinates are either all the same or all different, and their y coordinates are either all the same or all different. Find the number of combinations of 4 distinct lattice points with x and y coordinates between 1 and 3, inclusive, such that no 3 of them form a *set*.

Proposed by Jordan Lefkowitz

Solution by Tanishq Pauskar
Notice that for any 2 points on a 3×3 lattice grid, there is exactly one point that forms a set with them. For every pair of 2 points chosen, call the unique point that forms a set with them "blacklisted"

Claim: In a triplet of 3 points, none of the 3 pairs of points can blacklist the same point.

Proof: Suppose the points are A, B, and C. For the sake of contradiction, assume (A, B) and (B, C) blacklist the same point X. Then the pair of points X and C must blacklist 2 points, A and B which is a contradiction. By symmetry, the same applies to all of the other pairs.

We will choose points 1 at a time:

There are 9 ways to choose the first point
There are $9 - 1 = 8$ ways to choose a second point distinct from the first point.
There are $9 - 2 - \binom{2}{2} = 6$ ways to choose the third point distinct from the first and second point ($\binom{2}{2}$ subtracts the number of blacklisted points)
There are $9 - 3 - \binom{3}{2} = 3$ ways to choose the fourth point distinct from the first, second, or third point ($\binom{3}{2}$ subtracts the number of blacklisted points)

Because the points can be chosen in any order, the answer is

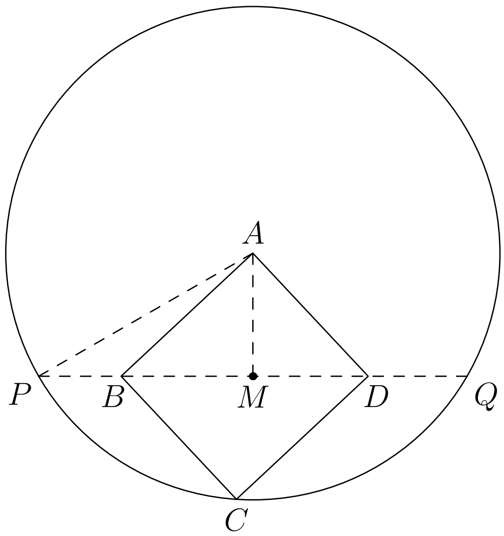
$$\frac{9 \cdot 8 \cdot 6 \cdot 3}{4!} = \boxed{54}$$

Problem 6

Let $ABCD$ be a rectangle and let ω be the circle with center A that passes through C. If line BD intersects ω at points P and Q so that $PB < PD$. Given that $PB = 10$ and $DQ = 12$, the area of $ABCD$ can be written as $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

Proposed by Tanishq Pauskar and Aarush Khare

Solution 1 by Aarush Khare



Let R be the radius of the circle. Consider the sum of the powers of B and D wrt ω. On one hand, this is

$$\begin{aligned} \text{pow}_\omega(B) + \text{pow}_\omega(D) &= PB \cdot BQ + PD \cdot DQ \\ &= PB \cdot (BD + DQ) + (PB + BD) \cdot DQ \\ &= 10(R + 12) + 12(R + 10) \end{aligned}$$

since $BD = R$. However, we also know that

$$\begin{aligned} \text{pow}_\omega(B) + \text{pow}_\omega(D) &= (R^2 - AB^2) + (R^2 - AD)^2 \\ &= 2R^2 - BD^2 \\ &= R^2. \end{aligned}$$

Therefore, we have

$$R = 10(R + 12) + 12(R + 10) \iff (R - 30)(R + 8) = 0.$$

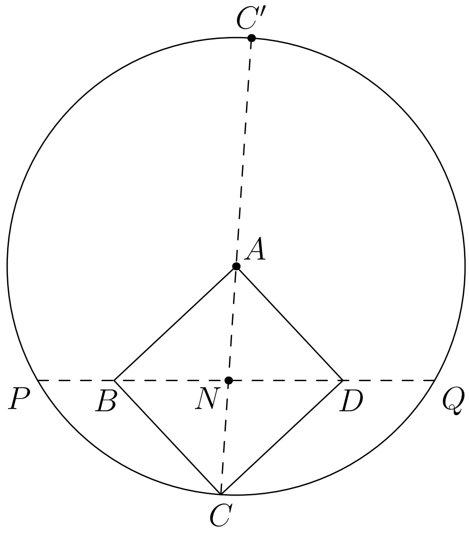
Discarding the negative solution, we get $R = 30$.

Now, drop the altitude from A to PQ to point M. We know that M is the midpoint of PQ, so $PM = \frac{10+30+12}{2} = 26$. Then

$$AM = \sqrt{AP^2 - PM^2} = \sqrt{30^2 - 26^2} = 4\sqrt{14},$$

so by $\frac{1}{2}bh$, the area of the rectangle is $2 \cdot \frac{1}{2} \cdot 4\sqrt{14} \cdot 30 = 120\sqrt{14} \Rightarrow \boxed{134}$.

Solution 2 by Govind Velamoor



Let N be the center of rectangle $ABCD$. By PoP on N, we have

$$NC \cdot NC' = NP \cdot NQ$$
$$\frac{R}{2} \cdot \frac{3R}{2} = \left(\frac{R}{2} + 10\right) \cdot \left(\frac{R}{2} + 12\right)$$

which simplifies to $(R - 30)(R + 8) = 0$, implying $R = 30$. We then finish the same as above.

Problem 7

Let $P(n)$ be a quadratic polynomial with integer coefficients. Jordan finds that, for all positive integers $n \geq 3$, every term of the sequence

$$P(n),\ P(P(n)),\ P(P(P(n))),\ \cdots$$

is a positive integer relatively prime to n. However, any two consecutive terms of this sequence sum to a multiple of n. Find $P(10)$.

Proposed by Aarush Khare

Solution by Aarush Khare

Our first claim is that the constant term of P is either 1 or -1. Indeed, the constant term cannot be 0 for obvious reasons, and it cannot be some other integer c, as then $P(c)$ would contain a factor of c.

Now, we similarly find that the numbers $P(P(n)), P(P(P(n))), \cdots$ must all have constant terms that are 1 or -1. Moreover, by the second condition, these constant terms must alternate between 1 and -1 in order for the sum of consecutive terms to be multiples of n. We claim that, for all k, if c is the constant term for P^k, then the constant term for P^{k+1} is $P(c)$. Indeed, note that the constant term of a P^{k+1} is just $P^{k+1}(0)$, which is equal to $P(P^k(0)) = P(c)$ as desired.

Now, since the constant terms alternate between ± 1, we must have $P(1) = -1$ and $P(-1) = 1$. We have two cases:

Case 1: $P(0) = 1$.
In this case, we find that $P(x) = -x^2 - x + 1$. However, this contradicts the fact that all terms of the sequence are positive integers.
Case 2: $P(0) = -1$.
In this case, we find that $P(x) = x^2 - x - 1$. This works, and is indeed the polynomial in question. The desired answer is $10^2 - 10 - 1 = \boxed{89}$.

Problem 8

For relatively prime positive integers x and y, define $f(x, y)$ to be the smallest positive multiple of x that is 1 more than a multiple of y. For relatively prime positive integers p and q,

$$f(p,\ q) + f(q,\ p) = 4321$$

Compute the remainder when the sum of all distinct possible values of $p + q$ is divided by 1000.

Proposed by Tanishq Pauskar, Jordan Lefkowitz

Solution by Aarush Khare

Let N be the value of $f(p, q) + f(q, p)$. Notice that $N \equiv 1 \pmod{p}$ and $N \equiv 1 \pmod{q}$, so

$$N \equiv 1 \pmod{pq}.$$

Moreover, we have $f(p, q) \leq pq$ and $f(q, p) \leq pq$, so $N \leq 2pq$. This implies that $N = pq + 1$. Then p and q are relatively prime, and satisfy

$$pq = 4320 = 2^5 \cdot 3^3 \cdot 5.$$

Therefore, the sum of all possible values of $p + q$ is $(1 + 2^5)(1 + 3^3)(1 + 5) = 5\boxed{544}$.

Note: for this problem, many people interpreted it as "$f(p, q)$ is the smallest positive multiple p that is one more than a **positive** multiple of q". This interpretation leads to the exclusion of the pair $(p, q) = (1, 4320)$, and leads to the answer 223. While this was not intended, we have decided that people who submitted 223 solved the problem just as much as the people who submitted 544 did, so we are accepting both $\boxed{223}$ and $\boxed{544}$ for this problem.

Problem 9

Define a sequence of rational numbers, $a_0, a_1, a_2, \cdots$ so that $a_{k+1} = |1 - \frac{1}{a_k}|$. Find the number of possible values of a_0 so that $a_{12} = 0$.

Proposed by Govind Velamoor, Tanishq Pauskar

Solution by Govind Velamoor
We work backwards. Notice that for any a_n,

$$a_n = 1 - \frac{1}{a_{n-1}}$$

and

$$-a_n = 1 - \frac{1}{a_{n-1}}$$

Case 1: $a_n > 1$
Consider the first equation. In this case, the solution for a_{n-1} will always be negative. Negative numbers cannot be in this sequence unless they are a_0, because every next number is defined as the absolute value of a number. In the second equation, we will always get a positive result which is less than 1.

Case 2: $a_n < 1$
From the first equation, we will get a positive solution greater than 1 and from the second we will get a positive solution less than 1.

Case 3: $a_n = 1$
We will always get $a_{n-1} = \frac{1}{2}$. Additionally, it is impossible for any a_n not a_{11} to be 1, because that would mean that $a_{n+1} = 0$, which is impossible. So, for any n such that $2 \le n \le 11$, it is true that if $a_n > 1$, then there is only one valid solution for a_{n-1} and $a_{n-1} < 1$, and if $0 < a_n < 1$, there are two solutions for a_{n-1}, one satisfying $a_{n-1} > 1$ and the other satisfying $0 < a_{n-1} < 1$.

We know that $a_{12} = 0$, so $a_{11} = 1$ and $a_{10} = \frac{1}{2}$. Now, we can use a recursion to find the number of values of a_n that are greater and less than 1. Let g_n and l_n be the number of valid solutions that are greater than 1 and less than 1 for a_n (where $1 \le n \le 11$). We know that $g_{n-1} = l_n$ and $l_{n-1} = l_n + g_n$. So, $l_{n-1} = l_n + l_{n+1}$. Because $l_{11} = 0$ and $l_{10} = 1$, this is the Fibonacci sequence in reverse! The sequence for g is the same, but lagging one term behind l. Because $l_{10} = 1$ and $g_{10} = 0$, we can compute that $l_1 = 55$ (the 10th Fibonacci number) and $g_1 = 34$ (the 9th Fibonacci number). So, a_1 has $55 + 34 = 89$ total solutions. Remembering that a_0 can be both positive and negative, we double this to find our answer: $89 \cdot 2 = \boxed{178}$.

Problem 10

Juan chooses a random divisor of 2310 and writes it on a blackboard. He repeats this process indefinitely, and stops when two numbers on the board are not relatively prime. Let N be the number of ways in which Juan stops after 4 numbers are written on the board. Compute the remainder when N is divided by 1000.

Proposed by Tanishq Pauskar

Solution by Tanishq Pauskar
Notice that the number 2310 can be expressed as the product $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$

Claim: There are $(n+1)^5$ ways to choose an ordered pair of n relatively prime numbers $(a_1, a_2, \ldots, a_n)$

Proof: Each prime divisor can be assigned to one of the n numbers or nothing. Because there are 5 prime divisors, it is $(n+1)^5$

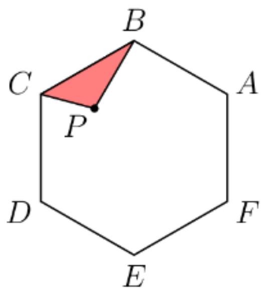
For Juan to choose a 4th number, he has to choose 3 relatively prime numbers first. Thus, there are $(3+1)^5 \cdot 32 = 8^5$ ways that he can end up choosing a $4th$ number. (There are 32 ways of choosing a divisor of 2310 for the 4th number)

It suffices to subtract the number of ways that he chooses 4 relatively prime numbers. (Juan will need to choose a 5th number if he chooses 4 relatively prime numbers). This is $(4+1)^5 = 5^5$.

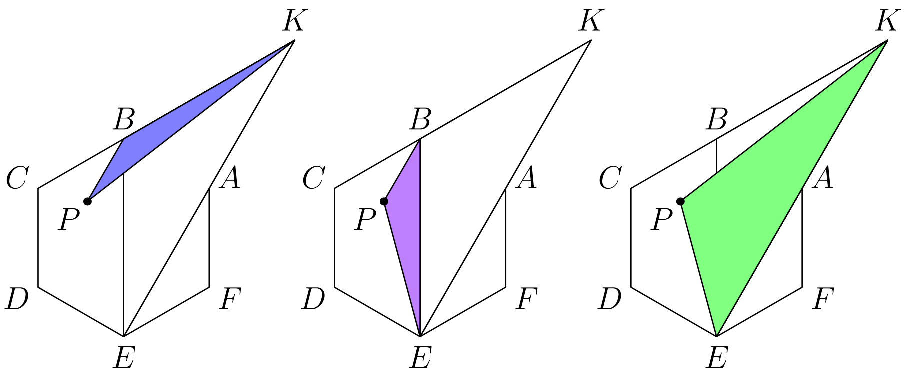
Thus, our answer is $8^5 - 5^5 = 29\boxed{643}$

Problem 11

A point P is positioned inside regular hexagon $ABCDEF$ so that $CP < AP$. Triangles BPC, BPE, and APE have areas 7, 12, and 28, respectively. Find the area of the hexagon.
Proposed by Aarush Khare



Solution by Aarush Khare
Define point K to be the intersection of BC and AE. Consider the following three areas:



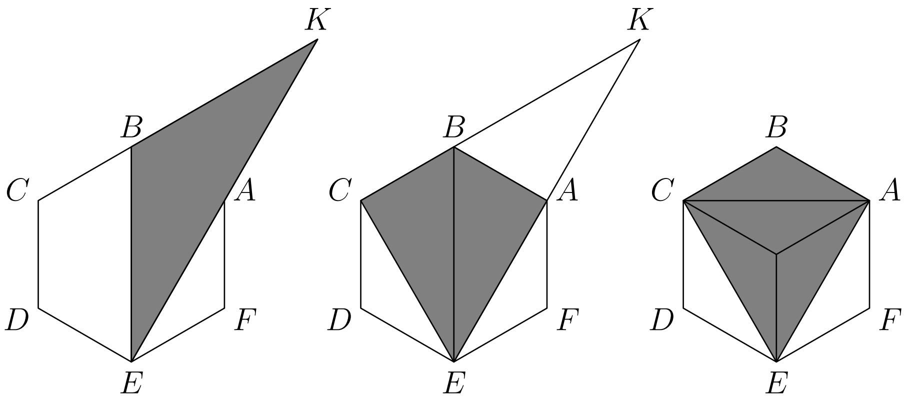
As we see from the diagram,

$$[BEK] = [PBK] + [PKE] - [PEB].$$

However, a simple length chase yields $BK = 2 \cdot CB$ and $KE = 2 \cdot EA$, so

$$[PBK] = 2 \cdot [BPC] = 14$$
$$[PKE] = 2 \cdot [APE] = 56$$
$$[PEB] = [BPE] = 12$$

which implies that $[BEK] = 14 + 56 - 12 = 58$. Now, to finish, notice that triangles BAK, BAE, and BCE are congruent, so the shaded regions in all of the following diagrams are the same



Thus, by the last diagram, the shaded region is $\frac{2}{3}$ of the area of the hexagon, making the final answer $\frac{3}{2} \cdot 58 = \boxed{87}$.

Problem 12

There exist quadratic functions $P(x)$ and $Q(x)$ with real coefficients. Each of the three equations

$$P(x) = Q(x)$$
$$P(x) = 2Q(x)$$
$$P(x) = 3Q(x)$$

has exactly 1 solution. Given that the solutions to these three equations are $x = 20$, 22, and n for a real number n, find the remainder when the product of all possible values of n is divided by 1000.

Proposed by Tanishq Pauskar and Aarush Khare

Solution by Aarush Khare
Consider the three polynomials $P - Q$, $P - 2Q$, $P - 3Q$. We claim that one of these is linear.
Let the leading coefficients of P and Q be a and b, respectively. Then $P - Q$, $P - 2Q$, $P - 3Q$ have leading coefficients $a - b$, $a - 2b$, and $a - 3b$, respectively. Assuming that all three of them are quadratic, this implies that

$$P - Q = (a - b)(x - 20)^2$$
$$P - 2Q = (a - 2b)(x - 22)^2$$
$$P - 3Q = (a - 3b)(x - n)^2$$

By the first two equations, we get

$$Q = (a - b)(x - 20)^2 - (a - 2b)(x - 22)^2,$$

and by the last two equations we get

$$Q = (a - 2b)(x - 22)^2 - (a - 3b)(x - n)^2.$$

Then

$$(a - 3b)(x - n)^2 = 2(a - 2b)(x - 22)^2 - (a - b)(x - 20)^2$$

Plugging in $x = 20$ and $x = 22$ yields

$$(20 - n)^2 = \frac{8(a - 2b)}{a - 3b}$$
$$(22 - n)^2 = \frac{-4(a - b)}{a - 3b}$$

and adding these two yields

$$(20 - n)^2 + (22 - n)^2 = 4$$

which implies that either $n = 20$ or $n = 22$. However, if $n = 20$, the first equation and last equation together imply that P and Q are both multiples of $(x - 20)^2$, which means the second equation cannot have a unique solution of 22. Similarly, $n = 22$ fails, so at least one of the three polynomials must be linear.

We now have three cases.
Case 1: $P - Q$ is linear.
In this case, we have $a = b$. Then

$$P - Q = c(x - 20)$$
$$P - 2Q = -a(x - 22)^2$$
$$P - 3Q = -2a(x - n)^2$$

By the first two equations,

$$Q = a(x-22)^2 + c(x-20)$$

and by the last two

$$Q = 2a(x-n)^2 - a(x-22)^2.$$

Then

$$2a(x-n)^2 = 2a(x-22)^2 + c(x-20)$$

so plugging in $x = 20$ yields $(20-n)^2 = 4$ or $n = 18$ (one can check that $n = 22$ doesn't work).

Case 2: $P - 2Q$ is linear.
In this case, we have $a = 2b$. Then

$$\begin{aligned} P - Q &= b(x-20)^2 \\ P - 2Q &= c(x-22) \\ P - 3Q &= -b(x-n)^2 \end{aligned}$$

By the first two equations,

$$Q = b(x-20)^2 - c(x-22)$$

and by the last two

$$Q = b(x-n)^2 + c(x-22).$$

Then

$$b(x-n)^2 = b(x-20)^2 - 2c(x-22)$$

so plugging in $x = 22$ yields $(22-n)^2 = 4$ or $n = 24$ (one can check that $n = 20$ doesn't work).

Case 3: $P - 3Q$ is linear.
In this case, we have $a = 3b$. Then

$$\begin{aligned} P - Q &= 2b(x-20)^2 \\ P - 2Q &= b(x-22)^2 \\ P - 3Q &= c(x-n) \end{aligned}$$

By the first two equations,

$$Q = 2b(x-20)^2 - b(x-22)^2$$

and by the last two

$$Q = b(x-22)^2 - c(x-n).$$

Then

$$c(x-n) = 2b(x-22)^2 - 2b(x-20)^2 = 2b(-2)(2x-42).$$

Both sides of the equation are linear, so their common solution must be 21, implying that $n = 21$.
In conclusion, the product of the possible values of n is $18 \cdot 21 \cdot 24 = \boxed{9\,072}$.

Problem 13

All the vertices of the triangle ABC lie on the graph $y = \frac{1}{x}$ on a cartesian plane. If its orthocenter lies on the point $(-1, -1)$ and its centroid lies on the point $(7, 9)$, the length of its circumradius can be expressed as $\sqrt{m}$. Find m.

Proposed by Tanishq Pauskar, Govind Velamoor

Solution by Tanishq Pauskar
We will use the well known formula $R = \frac{xyz}{4s}$ where x, y, z are the side lengths and s is the area of the triangle.

Suppose the points are $A = \left(a,\ \frac{1}{a}\right)$, $B = \left(b,\ \frac{1}{b}\right)$, $C = \left(c,\ \frac{1}{c}\right)$.

Note that the Centroid condition implies $a + b + c = 21$ and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 27$

Claim: $abc = 1$
Proof: The slope of BC is $\frac{\frac{1}{c} - \frac{1}{b}}{c-b} = -\frac{1}{bc}$ so the slope of the altitude from A to BC is bc. The equation of the altitude is $bc(x - a) = y - \frac{1}{a}$ Similarly, the slope from B to AC is ac. Thus, the equation of the altitude is $ac(x - b) = y - \frac{1}{b}$

Equating the two equations, we get $x = -\frac{1}{abc}$ and $y = -abc$. This is the point of the orthocenter, implying $abc = 1$

Step 1 (Finding xyz)
Using Pythagorean theorem, we get that

$$AB = \sqrt{(a-b)^2 + \left(\tfrac{1}{a} - \tfrac{1}{b}\right)^2} = |a - b|\sqrt{1 + (\tfrac{1}{ab})^2} = (a - b)\sqrt{1 + c^2}$$

Similarly, we get

$$BC = |b - c|\sqrt{1 + a^2}$$

and

$$AC = |c - a|\sqrt{1 + b^2}$$

Thus, $xyz = |(a - b)(b - c)(c - a)|\sqrt{(1 + a^2)(1 + b^2)(1 + c^2)}$

Step 2 (Finding $4s$)
Using shoelace formula, we get

$$\begin{aligned} 2s &= |(\tfrac{a}{b} + \tfrac{b}{c} + \tfrac{c}{a} - \tfrac{a}{c} - \tfrac{c}{b} - \tfrac{b}{a})| \\ &= |\tfrac{(a-b)(b-c)(c-a)}{abc}| \\ &= |(a - b)(b - c)(c - a)| \end{aligned}$$

Step 3 (Finish)

$$\frac{xyz}{4s} = \frac{|(a-b)(b-c)(c-a)|\sqrt{(1+a^2)(1+b^2)(1+c^2)}}{2|(a-b)(b-c)(c-a)|}$$
$$= \frac{\sqrt{(1+a^2)(1+b^2)(1+c^2)}}{2}$$
$$= \frac{\sqrt{1+a^2+b^2+c^2+a^2b^2+b^2c^2+a^2c^2+a^2b^2c^2}}{2}$$

Note that

$$a^2+b^2+c^2 = (a+b+c)^2 - 2(ab+bc+ac) = (a+b+c)^2 - \frac{2(ab+bc+ac)}{abc}$$
$$= (a+b+c)^2 - 2(\tfrac{1}{a}+\tfrac{1}{b}+\tfrac{1}{c}) = 387$$

Similarly

$$(ab)^2+(ac)^2+(bc)^2 = (\tfrac{1}{a})^2+(\tfrac{1}{b})^2+(\tfrac{1}{c})^2$$
$$= (\tfrac{1}{a}+\tfrac{1}{b}+\tfrac{1}{c})^2 - 2(\tfrac{1}{bc}+\tfrac{1}{ac}+\tfrac{1}{ab})$$
$$= (\tfrac{1}{a}+\tfrac{1}{b}+\tfrac{1}{c})^2 - 2(a+b+c) = 687$$

and

$$a^2b^2c^2 = 1$$

Thus,

$$\frac{\sqrt{1+a^2+b^2+c^2+a^2b^2+b^2c^2+a^2c^2+a^2b^2c^2}}{2}$$
$$= \frac{\sqrt{1+387+687+1}}{2} = \sqrt{\boxed{269}}$$

Problem 14

Bob has a shuffled deck of 12 cards each labelled a distinct integer from $1-12$. Every minute, he randomly chooses 2 from the deck, records their product on a board, and then removes them from the deck. After there are no more cards in the deck, the probability that all of the numbers on the Bob's board are at most 60 can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Find $m+n$.

Proposed by Tanishq Pauskar

Solution by Tanishq Pauskar
For all products to be at most 60,

1. The number 12 has to pair with a number ≤ 5
2. The number 11 has to pair with a number ≤ 5
3. The number 10 has to pair with a number ≤ 6
4. The number 9 has to pair with a number ≤ 7
5. The number 8 has to pair with a number ≤ 7

We will attempt to find the number of combinations of pairs with the numbers $9-12$

12: Can be with a number $1-5$. (5 ways)

11: Can be a number $1-5$ that is not paired with 12 (4 ways)

10: Can be with a number $1-6$ that is not paired with 12 or 11 (4 ways)

9: Can be with a number $1-6$ that is not paired with $12,\ 11$ or 10 (3 ways)

8: Can be paired with a number $1-7$ that is not paired with $12,\ 11,\ 10,$ or 9 (3 ways)

After these pairings, there exist two cards left unpaired ≤ 7 remaining. Thus, the numbers can be paired because the product of them will be ≤ 60 (1 way). We therefore have $5\cdot4\cdot4\cdot3\cdot3\cdot1 = 720$ possible combinations of pairings which satisfy the desired condition. There are $\frac{\binom{12}{2}\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}}{6!} = 10395$ total ways to pair the numbers up with no restriction. Thus, the probability is $\frac{720}{10395} = \frac{16}{231}$ which means the answer is $16+231 = \boxed{247}$

Problem 15

Let ABC be an isosceles triangle with $AB = BC = 70$. Circles ω_1 and ω_2 are externally tangent at point T, and are tangent to AB and BC at A and C respectively. The center of ω_1 is equidistant from B and C. The median from C to AB intersects ω_2 at $E \neq C$. Given that $AT = 52$, find EC.
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Solution by Govind Velamoor
Let M be the midpoint of AB. Because tangents BA and BC have equal length, the power of B WRT ω_1 and ω_2 is equal, so B lies on the radical axis of ω_1, ω_2. Call the center of ω_1 J.

Let ω_3 be the circle with radius 0 centered at B, and let r be the radius of ω_1. Now consider the power of B. Because B lies on the radical axis of ω_1, ω_2, we know that $BJ^2 - r^2 = CB^2$. But we know that $BJ = CJ$, so $CJ^2 - r^2 = CB^2$. The power of C WRT ω_3 is CB^2, and the power of C WRT ω_1 is $CJ^2 - r^2$. Because we know these two quantities are equal, C lies on the radical axis of ω_1, ω_3. But since M is the midpoint of AB, its power WRT ω_1 and ω_3 is equal as well, implying that CM is the radical axis of ω_1 and ω_3.

Now construct TB and let it intersect MC at R. Since TB is the radical axis of ω_1, ω_2, R is the radical center of $\omega_1, \omega_2, \omega_3$. This means that $BR = RT$, so MR is a midsegment of $\triangle BAT$, and $MR = \frac{AT}{2} = 26$. Let the altitude from B to AT intersect MR at F and AT at G. Because ABT is isosceles (by equal tangents AB and TB), we can compute $BF = \frac{BG}{2} = \frac{\sqrt{AB^2 - \left(\frac{AT}{2}\right)^2}}{2} = 4\sqrt{66}$

Also, $BC = 70$, so

$$FC = \sqrt{BC^2 - BF^2} = 62$$

. $FR = \frac{MR}{2} = 13$, so $RC = 49$. Now, power of a point on R WRT ω_2 yields $RT^2 = RE \cdot RC$. $RE = \frac{35^2}{49} = 25$. Finally, we have $EC = RC - RE = \boxed{24}$.