IMTC Division B Solutions

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Division B Answer Key

1	105	6	76	11	464
2	625	7	22	12	56
3	26	8	384	13	73
4	162	9	126	14	576
5	24	10	65	15	134

Problem 1

At the Math Aquarium, there are 70 more Mathaladons than Angler Fish. Given that the number of Mathaladons is 3 times the number of Angler Fish, how many Mathaladons are there?

Proposed by Tanishq Pauskar

Solution by Aarush Khare
Let x be the number of Angler Fish. Then we have the equation $3x = x + 70$, implying that $x = 35$. Then the answer is $3x = \boxed{105}$.

Problem 2

Jack is thinking of a positive integer with 3 distinct digits. Given that its units digit is a 5 and it can be expressed as the square of an integer, what number is Jack thinking of?

Proposed by Govind Velamoor

Solution by Govind Velamoor
The only 3-digit square numbers that end in 5s are $15^2 = 225$ and $25^2 = 625$. The number 225 has two 2s, so all its digits are not distinct, and it can't be Jack's number. Therefore, his number is $\boxed{625}$.

Problem 3

At IMTC, it is true that in X days, 4 people can write 24 problems, and in 17 days, Y people can write 54 problems. Given that $\frac{X}{Y}$ can be written in simplest form as $\frac{m}{n}$, find $m+n$.

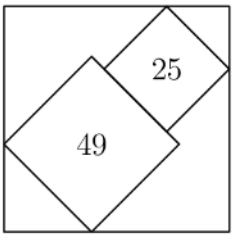
Proposed by Govind Velamoor

Solution by Govind Velamoor

If in X days 4 people can write 24 problems, then it is also true that in X days Y people can write $6Y$ problems. We also know that in 17 days, Y people can write 54 problems, so $\frac{X}{6Y} = \frac{17}{54}$. Then $\frac{X}{Y} = \frac{17}{9}$, so our answer is $17 + 9 = \boxed{26}$.

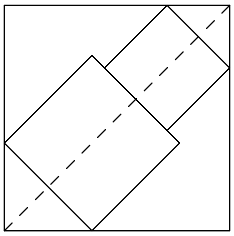
Problem 4

Two squares with areas 49 and 25 are inscribed inside a larger square in such a way that the centers of the smaller squares lie on a diagonal of the larger square. Find the area of the larger square.



Proposed by Aarush Khare

Solution by Aarush Khare



The two squares have side lengths 5 and 7. Notice that the two triangles in the corners are isosceles right triangles. Therefore, the length of the dashed segment is $\frac{5}{2} + 5 + 7 + \frac{7}{2} = 18$. Then the side length of the large square is $9\sqrt{2}$, making the answer $(9\sqrt{2})^2 = \boxed{162}$.

Problem 5

Bob has a deck of 6 cards labeled with a distinct integer from 1 to 6. Every second, he gives Alice a card, upon which Alice makes a statement about the sum of her cards. She says the following, in order:

The sum of my numbers is prime.
The sum of my numbers is a perfect square.
The sum of my numbers is a perfect cube.
The sum of my numbers is my age.
The sum of my numbers is a perfect square.
The sum of my numbers is 21.

Find the sum of the possible values of Alice's age.

Proposed by Aarush Khare

Solution by Aarush Khare
We first look at the third statement. The only perfect cubes between 1 and 21 are 1 and 8. Clearly her sum after three cards can't be 1, so it must be 8. Then the only option for the perfect square in the second statement is 4, so the third number is 4, and the first two numbers must be 1 and 3 in some order. Then by the first statement, we get that the first three numbers are 3, 1, 4.

Now, the only perfect square between 8 and 21 is 16, so that must be her sum after five cards. Therefore, the fourth and fifth cards sum to 8, so they must be 2 and 6 in some order. Thus, her possible orders are

$$3, 1, 4, 2, 6, 5$$

$$3, 1, 4, 6, 2, 5$$

In the first case, her age is 10, and in the second, her age is 14, which yields an answer of $\boxed{24}$.

Problem 6

A three-digit positive integer is chosen at random. Then the probability that the product of its digits will be prime can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Proposed by Aarush Khare

Solution by Aarush Khare

A three-digit positive integer has a prime digit product if and only if its digits are 1, 1, p in some order, where p is one of 2, 3, 5, 7. There are 4 possible choices for p, and 3 ways to arrange our number after we make our choice. On the other hand, there are 900 three-digit numbers, so the desired probability is $\frac{12}{900} = \frac{1}{75} \Rightarrow \boxed{76}$.

Problem 7

Martha chooses a positive integer n for which the sum of the digits of n is the same as the product of the digits of n. She then adds 101 to her number, and finds that it still satisfies this property. Find n.

Proposed by Aarush Khare

Solution by Aarush Khare
Consider how adding 101 to Martha's number changes the digit product. Assume that there is a carry-over when performing this addition, and focus on the leftmost one. Since the digits of 101 are 0 or 1, this carry over must occur from addition between a 1 and a 9, which will create a 0 in the resulting number. However, this means that the product of the digits of the result will be 0, while the sum will clearly be a positive integer. This means that we cannot have any carry-overs.

Now, assume that Martha's number has digits a and c in the hundreds place and units place, respectively. Since there are no carry overs, after addition these digits become $a+1$ and $c+1$. Then the product of the digits increases by a factor of $\frac{a+1}{a} \cdot \frac{c+1}{c}$, while the sum of the digits increases by 2. If N was the product/sum of the digits before the change, we have the equation

$$\tfrac{a+1}{a} \cdot \tfrac{c+1}{c} \cdot N = N + 2.$$

The quantity $\frac{a+1}{a}$ is minimized when $a = 8$ (we have $a \neq 9$ because there are no carry-overs), and similarly with $\frac{c+1}{c}$. Therefore, we have

$$N + 2 = \frac{a+1}{a} \cdot \frac{c+1}{c} \cdot N \geq \frac{81}{64} N$$

which implies that

$$\frac{128}{17} > 7 \geq N.$$

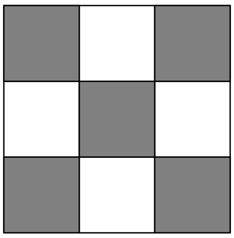
Now, the list of all numbers with digit sum at most 7 with the same digit sum and product is the one-digit numbers 1 to 7, the two-digit number 22, and the three-digit numbers $\{123, 132, 213, 231, 312, 321\}$. Out of these, only 22 and 123 differ by 101, so the answer is $\boxed{22}$.

Problem 8

The numbers 1 to 9 are placed in a 3×3 grid so that any two squares that share an edge sum to an odd number. Moreover, the two main diagonals of this grid both sum to multiples of 3. In how many ways can we do this?

Proposed by Aarush Khare

Solution by Aarush Khare



Color the board in a checkerboard fashion as shown above. In order to fulfill the first condition, the odd numbers must be placed in the shaded cells, as there are 5 odd numbers and 4 even numbers. Consider the remainder when the odd numbers are divided 3. The numbers 1, 3, 5, 7, 9 leave remainders of 1, 0, 2, 1, 0, respectively. Therefore, since we only have one 2, we should place 5 in the center cell, and place a 1 and a 0 in each diagonal.

For each of the two diagonals, there are 2 ways to decide which cell should be a 1 and which should be a 0. In addition, there are 4 ways to choose the numbers in the first diagonal (one of the pairs $(1, 3)$, $(1, 9)$, $(7, 3)$, $(7, 9)$). Finally, the even numbers can be placed in any way, so there are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways to place those. Our total is $2 \cdot 2 \cdot 4 \cdot 24 = \boxed{384}$.

Problem 9

Geoff chooses a three digits a, b, c from 1 to 9, and writes the seven numbers

$$a, b, c, a+b, b+c, c+a, a+b+c.$$

He finds that exactly five of these numbers are perfect squares. Find the product of the other two.

Proposed by Aarush Khare

Solution by Aarush Khare
First note that if none of the digits are perfect squares, at most 4 out of 7 of the given expressions can be perfect squares. Therefore, at least one of our digits is a perfect square. On the other hand, if all of our digits are perfect squares, then none of $a+b$, $b+c$, $c+a$ can be perfect squares, as a, b, c are all either 1, 4, or 9, and no two of these sum to a square. This still allows at most 4 out of 7 of the expressions to be perfect squares. Therefore, we either have 1 or 2 perfect squares amongst our digits.

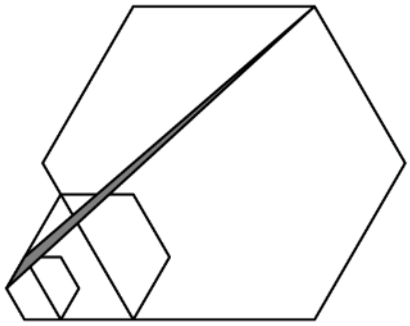
Now, assume that $a+b+c$ is not a perfect square. By the pigeonhole principle, either all of $a+b$, $b+c$, $c+a$ are perfect squares, or all of a, b, c are perfect squares. However, in both cases, we will find that two of our digits are perfect squares, say a and b, and that $a+b$ is also a perfect square, which is a contradiction as we showed above. Therefore $a+b+c$ must be a perfect square.

We know that one of $a+b$, $b+c$, $c+a$ must be a perfect square, as otherwise we only have four expressions left. Say that this square is $a+b$ (it doesn't matter which one we choose). Then we can't have both a and b perfect squares, so c must be a perfect square. Then both $a+b$ and $a+b+c$ are perfect square. The only possibility for this pair is $(a+b, a+b+c) = (16, 25)$, which implies that $c = 9$, and $(a, b) = (8, 8)$ or $(7, 9)$. Out of these two, only $(a, b, c) = (9, 9, 7)$ works.

This makes our seven numbers $7, 9, 9, 16, 16, 18, 25$, so the product of the two that aren't squares is $7 \cdot 18 = \boxed{126}$.

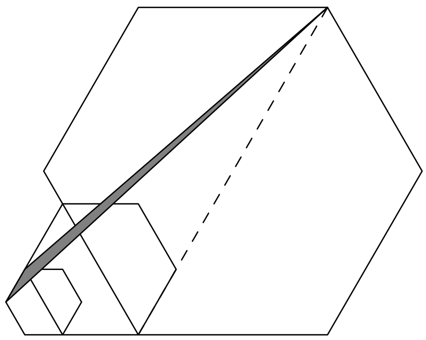
Problem 10

Three hexagons are drawn as shown. The shaded triangle has area 21. Then the area of the smallest hexagon can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

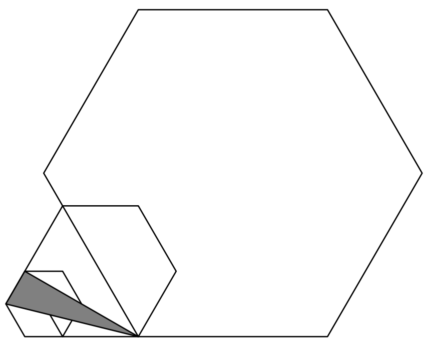


Proposed by Aarush Khare

Solution by Aarush Khare
Consider moving the point on the largest hexagon along the following dotted line



The line is parallel to the base of the triangle, so it doesn't change the area of the triangle. Therefore, the area of the shaded triangle is the same as the area of this triangle



If the side length of the smallest hexagon is s, this is a right triangle with legs of length s and $2s\sqrt{3}$. Equating the area of this triangle to 21 yields

$$2s^2\sqrt{3}\cdot\frac{1}{2}=21,$$

so $s^2=\frac{21}{\sqrt{3}}$. Then area the small hexagon can be broken up into 6 equilateral triangles with side length s, so the answer is $6\cdot\frac{s^2\sqrt{3}}{4}=\frac{63}{2}\Rightarrow\boxed{65}$.

Problem 11

Let $P(x)$ be a quadratic polynomial such that the product of the roots of P is 20. Real numbers a and b satisfy $a+b=22$ and $P(a)+P(b)=P(22)$. Find a^2+b^2.

Proposed by Tanishq Pauskar

Solution by Govind Velamoor
By Vieta's, we know the polynomial must be of the form $px^2+qx+20p$. Therefore,

$$P(a)+P(b)=pa^2+qa+20p+pb^2+qb+20p=p(a^2+b^2)+q(a+b)+40p.$$

This is the same as

$$P(22)=22^2p+22q+20p,$$

so

$$p(a^2+b^2)+q(a+b)+40p=22^2p+22q+20p,$$
$$p(a^2+b^2)+q(a+b)+20p=22^2p+22q.$$

Because we know that $a+b=22$,

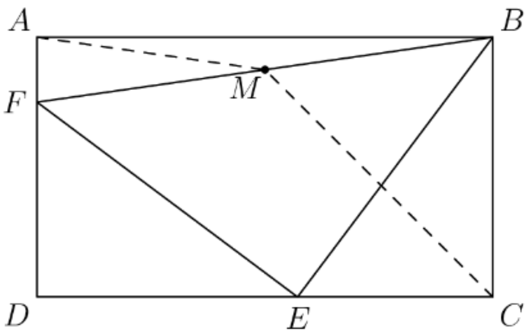
$$p(a^2+b^2)+22q+20p=22^2p+22q$$

$$p(a^2+b^2)+20p=22^2p$$
$$a^2+b^2+20=22^2.$$

So, we know $a^2+b^2=\boxed{464}$.

Problem 12

In rectangle $ABCD$, points E and F are chosen on sides CD and DA, respectively, such that triangle BEF is an isosceles right triangle with vertex E. Let M be the midpoint of BE. If $MA = 5$ and $MC = 7$, find the area of $ABCD$.



Proposed by Aarush Khare

Solution by Aarush Khare

First, notice that M is the circumcenter of $ABEF$, so $MA = MB = ME = MF = 5$, making $FE = EB = 5\sqrt{2}$. Then, note that the distance from M to DC is the average BC and FD, as M is the midpoint of BF.

Let $BC = a$ and $EC = b$. Notice that triangles FDE and ECB are congruent, so $DE = a$ and $FD = b$. Then the distance from M to DC is $\frac{a+b}{2}$, while the length of DC is $a + b$. This implies that triangle MDC is an isosceles right triangle, meaning $DC = 7\sqrt{2}$.

Thus, we have the system

$$a + b = 7\sqrt{2}$$
$$a^2 + b^2 = 50$$

which has solution $(a, b) = (4\sqrt{2}, 3\sqrt{2})$ (note that the opposite doesn't work as F lies outside the rectangle). Then the area of the rectanlge is $BC \cdot CD = 4\sqrt{2} \cdot 7\sqrt{2} = \boxed{56}$.

Problem 13

Let x and y be real numbers satisfying the system

$$2^x - 81y^2 = 0$$
$$3^x - 512y^3 = 0$$

Then y can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Proposed by Aarush Khare

Solution by Aarush Khare

By rearranging, we get

$$2^x = 81y^2$$
$$3^x = 512y^3.$$

Cubing the first equation and squaring the second, we get

$$8^x = 3^{12}y^6$$
$$9^x = 2^{18}y^6.$$

Dividing these two yields

$$\left(\frac{8}{9}\right)^x = \frac{3^{12}}{2^{18}} = \frac{9^6}{8^6} = \left(\frac{8}{9}\right)^{-6}$$

so $x = -6$. Plugging this into the first equation yields

$$y^2 = \frac{1}{3^4 \cdot 2^6}$$

which implies that $y = \frac{1}{72}$, which yields an answer of $\boxed{73}$.

Problem 14

Let $\mathcal{S}$ be the set of the first 8 prime numbers. For each subset $\mathcal{T}$ of $\mathcal{S}$, let $f(\mathcal{T})$ be the remainder when the product of the elements of $\mathcal{T}$ is divided by 6. Find the sum of $f(\mathcal{T})$ over all subsets of $\mathcal{S}$. Note: for the empty set, we define $f(\emptyset) = 1$.

Proposed by Tanishq Pauskar, Govind Velamoor

Solution by Tanishq Pauskar
Consider a subset $\mathbb{T}$ where $f(\mathbb{T}) \neq 0$ that does not contain the element 5.
Let $\mathbb{P}$ be the union of $\mathbb{T}$ and $\{5\}$. It is easy to see that

$$f(\mathbb{T}) + f(\mathbb{P}) = 6$$

This means across all subsets $\mathbb{X}$ where $f(\mathbb{X}) \neq 0$, the expected value of $f(\mathbb{X})$ is 3. The remainder is not equal to 0 when either 2 or 3 are not included in the set (which happens $\frac{3}{4}$ of the time). There are a total of $2^8 = 256$ subsets of $\mathbb{S}$. Thus, our answer is

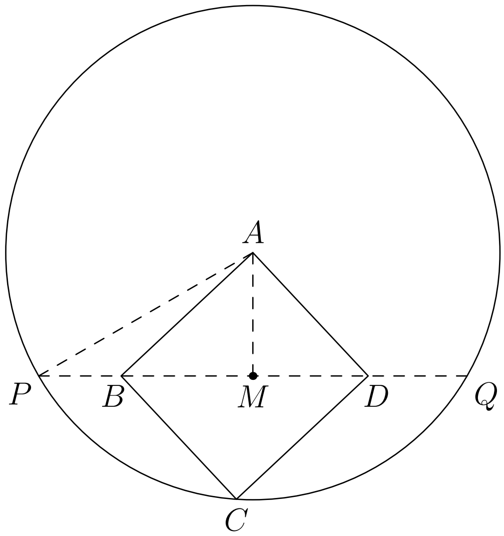
$$256 \cdot \frac{3}{4} \cdot 3 = \boxed{576}.$$

Problem 15

Let $ABCD$ be a rectangle and let ω be the circle with center A that passes through C. If line BD intersects ω at points P and Q satisfying $PB = 10$ and $DQ = 12$, compute the sum of all possible values of BD.

Proposed by Tanishq Pauskar and Aarush Khare

Solution 1 by Aarush Khare



Let R be the radius of the circle. Consider the sum of the powers of B and D wrt ω. On one hand, this is

$$\begin{aligned}\text{pow}_\omega(B) + \text{pow}_\omega(D) &= PB \cdot BQ + PD \cdot DQ \\ &= PB \cdot (BD + DQ) + (PB + BD) \cdot DQ \\ &= 10(R + 12) + 12(R + 10)\end{aligned}$$

since $BD = R$. However, we also know that

$$\begin{aligned}\text{pow}_\omega(B) + \text{pow}_\omega(D) &= (R^2 - AB^2) + (R^2 - AD)^2 \\ &= 2R^2 - BD^2 \\ &= R^2.\end{aligned}$$

Therefore, we have

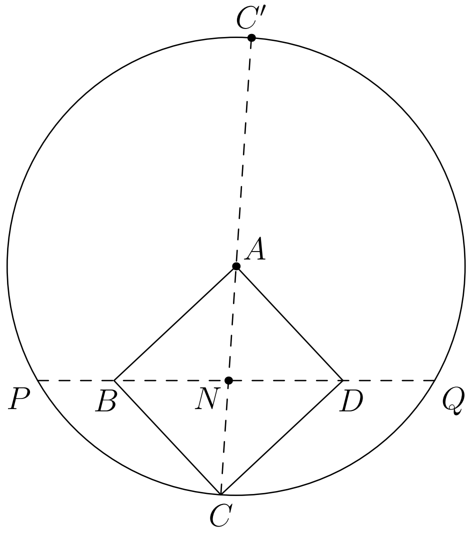
$$R = 10(R + 12) + 12(R + 10) \iff (R - 30)(R + 8) = 0.$$

Discarding the negative solution, we get $R = 30$.
Now, drop the altitude from A to PQ to point M. We know that M is the midpoint of PQ, so $PM = \frac{10+30+12}{2} = 26$. Then

$$AM = \sqrt{AP^2 - PM^2} = \sqrt{30^2 - 26^2} = 4\sqrt{14},$$

so by $\frac{1}{2}bh$, the area of the rectangle is $2 \cdot \frac{1}{2} \cdot 4\sqrt{14} \cdot 30 = 120\sqrt{14} \Rightarrow \boxed{134}$.

Solution 2 by Govind Velamoor



Let N be the center of rectangle $ABCD$. By PoP on N, we have

$$NC \cdot NC' = NP \cdot NQ$$
$$\frac{R}{2} \cdot \frac{3R}{2} = \left(\frac{R}{2} + 10\right) \cdot \left(\frac{R}{2} + 12\right)$$

which simplifies to $(R - 30)(R + 8) = 0$, implying $R = 30$. We then finish the same as Solution 1.