IMTC Division A

Rules

This is a fifteen question short answer exam, for which you will have 180 minutes to finish.
Each question has a nonnegative integer answer from 000 to 999. It does not matter if you add leading zeroes to your answers.
Questions are not necessarily ordered by difficulty, and diagrams are not necessarily drawn to scale.
For each question, you will receive 10 points for a correct answer and 0 points for an incorrect answer or if the question is left blank.
No aids are permitted other than scratch paper, graph paper, rulers, and writing instruments.
The use calculators, smartwatches, or computing devices is prohibited. No problems on the exam will require the use of a calculator.
Discussion about problems from this before the submission deadline is strictly prohibited.

Do not exit the tab or switch to another tab at any point during the exam. Doing so lowers the integrity of the exam and might result in a disqualification.

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Problem 1

Compute the number of positive integers $n > 1$ that satisfy

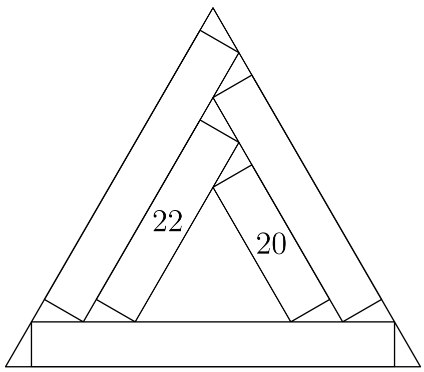
$$\lfloor \log_2 n \rfloor = \left\lfloor \frac{n}{100} \right\rfloor .$$

Problem 2

Find the unique three-digit prime number p with distinct digits $\underline{a}\ \underline{b}\ \underline{c}$ such that the last two digits of p^2 are $\underline{a}\ \underline{b}$ and the last digit of p^3 is $\underline{c}$.

Problem 3

Five rectangles with the same height are positioned in an equilateral triangle as shown. Two of the rectangles have areas 20 and 22, as indicated. Find the area of the triangle.



Problem 4

Alfred chooses a three digit positive integer. He then tells Beth, Carlos, and Daniel the hundreds, tens, and units digit of the number, respectively. With the information they are given, Beth, Carlos, and Daniel each find the probability that Alfred's number is less than 540. They tell their probabilities to Alfred in some order, and Alfred finds that they form an arithmetic sequence. How many numbers could Alfred have chosen?

Problem 5

Let ABC be a triangle with orthocenter H. Let point D to be the foot of the altitude from B to AC, and let point M to be the midpoint of AC. Given that $MD = 1$, $DH = 4$, and $HB = 8$, find the area of ABC.

Problem 6

An angle θ is randomly chosen from the interval $[0, 2\pi]$. The probability that $\sin(3\theta) > \sin(2\theta) > \sin(\theta)$ can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Find $m + n$.

Problem 7

Find the maximum possible value of $a + b + c$ over all ordered triples of positive integers (a, b, c) satisfying

$$\phi(a)^2 + \phi(b)^2 + \phi(c)^2 = 2021.$$

(Here, $\phi(x)$ denotes the number of positive integers less than or equal to x that are relatively prime to x.)

Problem 8

Jessica is writing a sequence a_i on a whiteboard. For all positive integers i, she defines

$$a_i = \sqrt{\frac{i + 2022}{i}}.$$

Let N be the sum of all i for which a_i is a rational number. Compute the remainder when N is divided by 1000.

Problem 9

Let $ABCD$ be an isosceles trapezoid with circumcenter O. Let E be the intersection of AC and BO. Similarly, let F be the intersection of BD and AO. Given that $AB = 5$, $CD = 8$, and $EF = 3$, the area of $ABCD$ can be written as $\frac{m}{n}$. Find $m + n$.

Problem 10

There are 10 distinguishable penguins numbered from 1 to 10 sitting in a row of 10 seats. One day, Fred adds 3 empty seats to the left of the row and then goes to sleep. Every subsequent day, one penguin moves from its current seat to an empty seat to the left. After many days, Fred returns, and finds that the penguins now occupy the 10 leftmost seats in the row. Let N be the number of possible seating arrangements the penguins could be in. Compute the remainder when N is divided by 1000.

Problem 11

Three points A, B, and C lie on the graph of $y = x^2$. Given that lines AB, BC, and CA contain points $(1, -7)$, $(1, 4)$, and $(1, 7)$ respectively, the area of ABC can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Find $m + n$.

Problem 12

Alice and Bob have a deck of cards with 7 cards labeled $1,\ 2,\ 3, \cdots, 7$. Every minute, they flip a fair coin. If it lands on heads, Alice takes a random card from the deck, and if it lands on tails, Bob takes a random card from the deck. This is done until no cards remain in the deck. The probability that after every minute, Alice has the largest card amongst the two can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Find m.

Problem 13

Let $ABCD$ be a parallelogram with $AB = 5$, $AD = 3$, and $BD = 7$. Let X be a point on the angle bisector of $\angle BAD$. Lines XB and CD meet at F, while lines XD and BC meet at E. Suppose that triangle XEF is equilateral. The smallest possible value of EF can be expressed as $a\sqrt{b} - c\sqrt{d}$, where a, b, c, d are positive integers and b and d are not divisible by the square of any prime. Find $a + b + c + d$.

Problem 14

For each positive integer x, let $f(x)$ denote the least positive integer r such that the remainder when x^r is divided by 127 is at most 2. Compute the remainder when $f(1) + f(2) + \cdots + f(126)$ is divided by 1000.

Problem 15

Let N be the number of ordered pairs (a, b) of positive integers such that

$$a + b = 4^{12}$$

and each digit from 0 to 3 occurs the same number of times in the base 4 representation of a as it does in the base 4 representation of b (leading zeroes do not count). Compute the remainder when N is divided by 1000.