IMTC Division B

Rules

This is a fifteen question short answer exam, for which you will have 75 minutes to finish.
Each question has a nonnegative integer answer from 000 to 999. It does not matter if you add leading zeroes to your answers.
Questions are not necessarily ordered by difficulty, and diagrams are not necessarily drawn to scale.
For the first 10 questions, you will receive 6 points for a correct answer and 0 points for an incorrect answer or if the question is left blank. For the last 5 questions, you will receive 8 points for a correct answer and 0 points for an incorrect answer or if the question is left blank.
No aids are permitted other than scratch paper, graph paper, rulers, and writing instruments.
The use calculators, smartwatches, or computing devices is prohibited. No problems on the exam will require the use of a calculator.
Discussion about problems from this before the submission deadline is strictly prohibited.

Do not exit the tab or switch to another tab at any point during the exam. Doing so lowers the integrity of the exam and might result in a disqualification.

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Problem 1

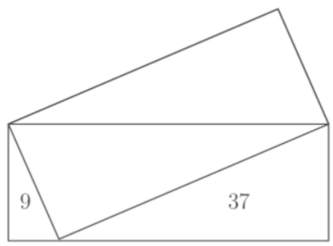
Positive integers x, y, z satisfy the system

$$\begin{aligned} x + y &= 8 \\ y + z &= 22 \\ z + 2x &= 23. \end{aligned}$$

Find xyz.

Problem 2

A figure is formed by two rectangles positioned as shown in the diagram below. The two indicated triangles have areas 9 and 37. Find the area of the entire figure.



Problem 3

Four real numbers a, b, c, d satisfy the equations

$$\frac{15a}{b} = \frac{24b}{c} = \frac{75c}{a} = d.$$

Find d.

Problem 4

There exist two positive integers that are equal to three times the product of their digits. Find their sum.

Problem 5

There are 9 balloons equally spaced around a circle. Every second, Bob will randomly pop a balloon that has not yet been popped, and the balloons adjacent to that balloon that are still inflated will also pop. Then the probability that every Bob will pop every balloon in exactly 3 seconds can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Note: This question was stated incorrectly on the contest in a way that made it unsolvable and the problem was voided. The corrected version is shown above.

Problem 6

Rectangle $ABCD$ is inscribed in circle ω. Point E lies on ω so that triangle ABE is equilateral. There exists a point F on segment CD such that the areas of triangles AEF and BEF are 20 and 22, respectively. Find the area of rectangle $ABCD$.

Problem 7

Compute the value of the How many ordered pairs of positive integers, (a, b), exist with $1 \leq a,\ b \leq 27$ such that $\frac{\gcd(3a,b)}{\gcd(a,3b)}$ is an integer?

Problem 8

Let N be the value of the sum

$$\frac{1}{\frac{1}{1}+\frac{1}{2020}}+\frac{1}{\frac{1}{2}+\frac{1}{2019}}+\frac{1}{\frac{1}{3}+\frac{1}{2018}}+\cdots+\frac{1}{\frac{1}{1010}+\frac{1}{1011}}.$$

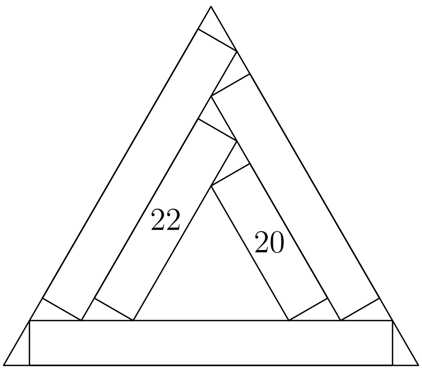
Compute the remainder when N is divided by 1000.

Problem 9

Find the unique three-digit prime number p with distinct digits $\underline{a}\,\underline{b}\,\underline{c}$ such that the last two digits of p^2 are $\underline{a}\,\underline{b}$ and the last digit of p^3 is $\underline{c}$.

Problem 10

Five rectangles with the same height are positioned in an equilateral triangle as shown. Two of the rectangles have areas 20 and 22, as indicated. Find the area of the triangle.



Problem 11

Let ABC be a triangle with orthocenter H. Let point D to be the foot of the altitude from B to AC, and let point M to be the midpoint of AC. Given that $MD = 1$, $DH = 4$, and $HB = 8$, find the area of ABC.

Problem 12

Let N be the number of ways can an ordered quintuplet of 5 not necessarily distinct subsets of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ be made such that there are exactly 3 elements in an odd number of subsets. Find the number of factors of N.

Problem 13

Jessica is writing a sequence a_i on a whiteboard. For all positive integers i, she defines

$$a_i = \sqrt{\frac{i + 2022}{i}}.$$

Let N be the sum of all i for which a_i is a rational number. Compute the remainder when N is divided by 1000.

Problem 14

Let $f(x) = x^{\lceil 52x/53 \rceil}$. Let $g(x)$ be difference of sum of the values of $f(x)$ for even positive numbers less than or equal to x, and the sum of the values of $f(x)$ for odd positive values less than or equal to x. There are exactly 4 values of m less than 250 such that $g(m) \equiv 0 \pmod{53}$. Find the sum of these 4 values.

Problem 15

Let $ABCD$ be an isosceles trapezoid with circumcenter O. Let E be the intersection of AC and BO. Similarly, let F be the intersection of BD and AO. Given that $AB = 5$, $CD = 8$, and $EF = 3$, the area of $ABCD$ can be written as $\frac{a}{b}$. Find $a + b$.