

IMTC Division A, Summer 2023

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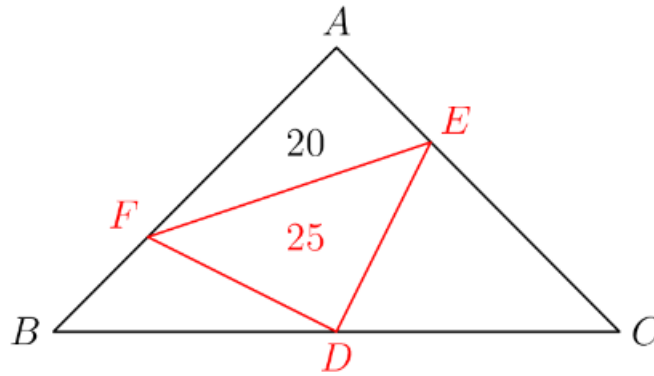


Problem 1

Suppose $P(x)$, $Q(x)$, and $P(x) + Q(x)$ are quadratic polynomials with vertices of $(4, 5)$, $(9, 15)$, $(8, 80)$ respectively. Find $P(2) + Q(3)$.

Problem 2

Let ABC and DEF be isosceles right triangles with $\angle A = \angle D = 90$. Triangle DEF is inscribed in ABC so that points D, E, F lie on segments BC, AC, AB respectively. Given that the area of AEF is 20 and the area of DEF is 25, find the area of ABC .



Problem 3

Matt is writing an infinite sequence starting with a positive integer a_1 . For $n \geq 2$, he forms a_n by removing the last digit of a_{n-1} , multiplying it by 2023, and adding it to the rest of the number. For example, if $a_1 = 24$, then a_2 would be $2 + 4 \cdot 2023 = 8094$. There exists a three digit prime p that divides every term of this sequence. Find p .

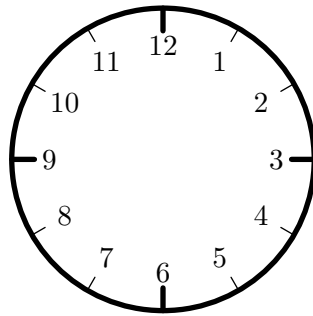


Problem 4

Let \mathbb{S} be a nonempty set and let $f(x)$ be the sum of the remainders when x is divided by all of the elements in \mathbb{S} . Given that $f(56) = f(57)$, find the sum of the possible values of $|\mathbb{S}|$.

Problem 5

On the analog clock shown, the numbers 1, 2, 3, 4, \dots 12 are attached in a clockwise manner. Every hour, a number is randomly chosen and removed from the clock. If the probability that at some point in time, the numbers 2 and 7 were adjacent to each other can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n , find m .

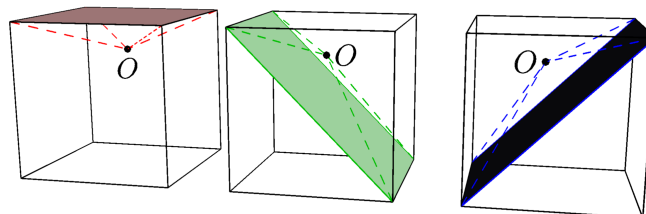


Problem 6

Let ABC be a triangle with incircle ω . The altitude from A to BC intersects ω at points E and F and intersects BC at point G so that F lies between E and G . Given that $AE = 5$, $EF = 15$, and $FG = 1$, find the perimeter of ABC .

Problem 7

Point O is positioned inside a cube so that it is closest to the top face of the cube. Three pyramids are drawn with their apex at point O , with their bases colored red, green, and blue. These three pyramids have volumes of 4, 6, and 7, respectively. Find the volume of the cube.





Problem 8

Let $P(x) = x^2 + 22x + 117$ and $Q(x)$ be another monic quadratic polynomial with real coefficients. Given that the equations $P(Q(x)) = 0$ and $Q(P(x)) = 0$ both have exactly 1 real solution, find $Q(23)$.

Problem 9

Suppose k is a real number so that there exists a non-zero polynomial $P(x)$ such that

$$P(x) = P(20)x^2 + P(k)x + P(23)$$

Find the product of the possible values of k

Problem 10

Scalene triangle ABC has incenter I . Let M and N be midpoints of AB and AC . If $IM = IN = 13$, and $BC = 48$, find the area of ABC .

Problem 11

Let X be the number of ordered pairs of monic polynomials with integer coefficients $(P(x), Q(x))$ such that

$$P(x) \cdot Q(x) = (x - 1)(x - 2)(x - 3) \dots (x - 17)$$

and the equation $\max(P(x), Q(x)) = 0$ has exactly 10 real solutions. Compute the remainder when X is divided by 1000.



Problem 12

For a real number x , let $\lfloor x \rfloor$ be the greatest integer less than or equal to x , and define $x - \lfloor x \rfloor$ to be the fractional part of x . For a positive integer n , the infinite sequence of numbers

$$\frac{n}{2}, \frac{n}{2^2}, \frac{n}{2^3}, \frac{n}{2^4}, \dots$$

is written on a whiteboard. Upon noticing the whiteboard, Bob sums the fractional parts of every term of the sequence and Joe sums the fractional part of every third term in the sequence (starting with $\frac{n}{2^3}$). Given that Bob's sum is twice Joe's sum, find the smallest possible value of n .

Problem 13

Suppose that on a cartesian plane, there exist points A and B lie on the graph of $y = x^2$, points C and D lie on the graph of $y = 2x^2$, and points E and F lie on the graph of $y = 3x^2$. If quadrilaterals $ACBD$ and $AEBF$ are non self-intersecting parallelograms with areas 7 and 6 respectively, the area of quadrilateral $CEDF$ can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n . Find $m + n$.

Problem 14

There are 21 students standing in a line. Every hour, a random pair of adjacent students have a fight, causing the losing student to exit the line. Given that each student has an equal chance of winning each fight, the probability that the final fight is between the third student from the left and the fourth student from the right in the initial line can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n . Find $m + n$.

Problem 15

Let ABC be an acute triangle and let X and Y be points on AB and AC respectively so that the circumcircle of triangle AXY is tangent to BC at the foot of the A -altitude. Suppose the B -altitude and C -altitude intersect XY at points P and Q so that $XP = 1$, $PQ = 2$, and $QY = 6$. Find the remainder when the square of the area of ABC is divided by 1000